ON PERTURBATIONS OF THE ISOMETRIC SEMIGROUP OF SHIFTS ON THE SEMIAXIS

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Abstract. We study perturbations $(\tilde{\tau}_t)_{t\geq 0}$ of the semigroup of shifts $(\tau_t)_{t\geq 0}$ on $L^2(\mathbb{R}_+)$ with the property that $\tilde{\tau}_t - \tau_t$ belongs to a certain Schatten–von Neumann class $\mathfrak{S}_p$ with $p \geq 1$. We show that, for the unitary component in the Wold–Kolmogorov decomposition of the cogenerator of the semigroup $(\tilde{\tau}_t)_{t\geq 0}$, *any singular* spectral type may be achieved by $\mathfrak{S}_1$ perturbations. We provide an explicit construction for a perturbation with a given spectral type based on the theory of model spaces of the Hardy space H^2. Also we show that we may obtain *any* prescribed spectral type for the unitary component of the perturbed semigroup by a perturbation from the class $\mathfrak{S}_p$ with $p > 1$.

Keywords. Semigroup of shifts, trace-class perturbation, Schatten–von Neumann ideals, Hardy space, inner function.

1. Introduction

Consider the isometric semigroup $(\tau_t)_{t\geq 0}$ of shifts on the space $L^2(\mathbb{R}_+)$,

$$(\tau_t f)(x) = \begin{cases} f(x-t), & x \geq t, \\ 0, & x < t, \end{cases} \qquad f \in L^2(\mathbb{R}_+).$$

In this paper we are concerned with perturbations $(\tilde{\tau}_t)$ of the semigroup (τ_t) satisfying the following properties:

$(\tilde{\tau}_t)_{t\geq 0}$ is a strongly continuous semigroup of isometric operators on $L^2(\mathbb{R}_+)$;

the difference $\tilde{\tau}_t - \tau_t$ belongs to a certain Schatten–von Neumann ideal $\mathfrak{S}_p$ for every $t > 0$.

The central problem considered in this paper is to describe all possible spectral types of perturbed isometric semigroups. The spectral type of a semigroup determines the semigroup uniquely up to the unitary equivalence; it is defined by the spectral type of the cogenerator of the group (see definition in §2). For $p = 1$ it follows from the stability of the absolutely continuous spectrum of the unitary dilation that the absolutely continuous parts of the cogenerators of the unitary dilations of the semigroups (τ_t) and $(\tilde{\tau}_t)$ are unitarily equivalent (see §2

The work is partially supported by Federal program 2.1.1/1662, by RFBR grant 08-01-00723, and by the President of Russian Federation grant NSH 2409.2008.1.

for details). The cogenerator of the semigroup (τ_t) is unitarily equivalent to the unilateral shift operator on the Hardy space H^2. Thus, for $p = 1$, our problem reduces to the description of all possible singular parts, and we show that *any singular* type may be realized by some semigroup $(\tilde{\tau}_t)_{t\geq 0}$. For $p = 2$ it was shown in [1, 2] that *any* spectral type of the unitary component is possible; for the case of a singular spectral measure a model of such perturbations was constructed. Here we show that analogous results are valid for all $p > 1$.

A possible motivation for our study is a connection with Markovian perturbations of the unitary group of shifts. The semigroup $(\tau_t)_{t\geq 0}$ is the restriction to $L^2(\mathbb{R}_+)$ of the unitary group of shifts $(\gamma_t)_{t\in\mathbb{R}}$, $(\gamma_t f)(x) = f(x-t)$, on the space $L^2(\mathbb{R})$ on the whole line (we identify the spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$ with the subspaces of functions from $L^2(\mathbb{R})$ identically equal to zero on the semiaxes $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively). Consider a perturbed unitary group $(\tilde{\gamma}_t)_{t\in\mathbb{R}}$ and assume that it has the so-called Markovian property, which means that the perturbed operators coincide with the unperturbed ones on the left semi-axis $\mathbb{R}_-$ as $t < 0$, i.e.,

$$\gamma_t f = \tilde{\gamma}_t f \quad \text{if} \quad f = 0 \ \text{ on } \ \mathbb{R}_+, \qquad t < 0. \tag{1}$$

As usual, the Markovian property may be interpreted in the sense that "the past does not depend on the future". Markovian perturbations with the additional property $\tilde{\gamma}_t - \gamma_t \in \mathfrak{S}_2$, $t \in \mathbb{R}$, were investigated by the first author (see, e.g., [15]) in connection with cocycle perturbations of the flow of Powers shifts [17].

Sometimes it will be convenient to work with the model in the Hardy space H^2 on the unit circle $\mathbb{T}$ which is unitarily equivalent to the original one (see §4). The semigroup $(\tau_t)_{t\geq 0}$ on $L^2(\mathbb{R}_+)$ is unitarily equivalent to the semigroup of operators on H^2 of multiplication by the functions φ_t, $t \geq 0$, where

$$\varphi_t(z) = \exp\left(t\frac{z+1}{z-1}\right). \tag{2}$$

In this case the cogenerator of the unperturbed semigroup is the unilateral shift S, i.e., the operator of multiplication by z in H^2. Denote by $\tilde{S}$ the cogenerator of the perturbed semigroup; then its elements are of the form $\varphi_t(\tilde{S})$.

Now we introduce a new parameter of our model of perturbations, namely, an inner function θ in the unit disk (a function $\theta \in H^2$ is said to be inner if $|\theta| = 1$ almost everywhere on $\mathbb{T}$). Consider the S-coinvariant subspace

$$K_\theta = H^2 \ominus \theta H^2.$$

For the theory the backward shift invariant subspaces, also called model subspaces, see [6, 21].

In what follows we are interested in perturbations $\tilde{S}$ of the shift operator S with the following properties:

(i) $\tilde{S}$ is an operator on H^2, diagonal with respect to the decomposition $H^2 = K_\theta \oplus \theta H^2$;

(ii) $\tilde{S}$ acts on θH^2 as multiplication by z;

(iii) the restriction of $\tilde{S}$ to K_θ is a unitary operator, and 1 is not its eigenvalue.

Conditions (i)–(iii) mean that $\tilde{S}$ is an isometry, whose unitary and completely non-unitary parts act on K_θ and θH^2, respectively. The number 1 does not belong to the point spectrum of $\tilde{S}$, and the isometric semigroup with cogenerator $\tilde{S}$ is well defined.

Now we state our main results. First of them shows that any singular summand V may be obtained by means of trace class perturbations of the isometric semigroup of shifts in the model in the unit circle.

Theorem 1.1. *Let V be a singular unitary operator of multiplicity $n \le \infty$ such that 1 is not an eigenvalue of V, and let $\varepsilon > 0$. Then there exist an inner function θ and an operator $\tilde{S}$ with the properties* (i)–(iii) *such that*

a) the restriction of $\tilde{S}$ to K_θ is unitarily equivalent to V;

b) $\operatorname{rank}(\tilde{S} - S) \le n$;

c) $||\tilde{S} - S||_{\mathfrak{S}_1} \le \varepsilon$;

d) $\varphi_t(\tilde{S}) - \varphi_t(S) \in \mathfrak{S}_1$ for all $t > 0$.

As an immediate consequence of Theorem 1.1, we obtain the following statement about perturbations of the semigroup of shifts on the semiaxis.

Theorem 1.2. *Let $(\tau_t)_{t\ge 0}$ be the semigroup of shifts on $L^2(\mathbb{R}_+)$.*

1) If $(\tilde{\tau}_t)$ is an isometric semigroup such that $\tilde{\tau}_t - \tau_t \in \mathfrak{S}_1$ for all $t \ge 0$, then the cogenerator of the semigroup $(\tilde{\tau}_t)$ is unitarily equivalent to the direct sum of the shift operator (of multiplicity one) and a unitary operator with a singular spectral measure.

2) For any singular unitary operator V with the only restriction that 1 does not belong to its point spectrum, there exists an isometric semigroup $(\tilde{\tau}_t)$ whose cogenerator is unitarily equivalent to the operator $S \oplus V$, and $\tilde{\tau}_t - \tau_t \in \mathfrak{S}_1$ for all $t \ge 0$.

As another corollary of Theorem 1.1 we show that for $p > 1$ an analogous statement is true without the assumption that the unitary operator V is singular.

Theorem 1.3. *Let V be a unitary operator such that 1 is not an eigenvalue of V. Then there exist an inner function θ and an operator $\tilde{S}$ satisfying* (i)–(iii)*, for which the restriction of $\tilde{S}$ to K_θ is unitarily equivalent to V and $\varphi_t(\tilde{S}) - \varphi_t(S) \in \mathfrak{S}_p$ for all $p > 1$.*

For the case $p = 2$, analogs of Theorems 1.1, 1.3 were obtained in [1, 2]. Here we also consider separately the results about the class $\mathfrak{S}_2$, since in terms

of our model their proofs are essentially simplified and one may expect that the conditions obtained are sharp. The new results of this paper are connected with more narrow classes, i.e., $\mathfrak{S}_p$ with $p < 2$, and, in the first place, with $p = 1$.

In [16, 1] the construction of the operator $\tilde{S}$ was based on the triangulation studied by Ahern and Clark [14]. Instead, we use Clark's construction [18], which establishes an isometric identification of the space K_θ with a space $L^2(\mu)$ for a certain singular measure μ on the unit circle. This approach allows us to relate approximation properties of the difference $\varphi_t(\tilde{S}) - \varphi_t(S)$ with differential properties of μ.

The situation becomes essentially different when we pass to the "natural" unitary dilation of the perturbed semigroup $\tilde{S}$ with properties (i)–(iii) (see §2). Let U be the bilateral shift on $L^2(\mathbb{T})$ and let $\tilde{U}$ be a dilation of $\tilde{S}$ with the Markovian property, i.e., such that $\tilde{U}^*$ coincides with U^* on H^2_-. It turns out that then $\varphi_t(\tilde{U}) - \varphi_t(U)$ will never belong to $\mathfrak{S}_1$. However, for any unitary operator V there exists a construction of $\tilde{S}$ with the properties (i)–(iii) and a Markovian unitary dilation $\tilde{U}$ of $\tilde{S}$ such that the restriction of $\tilde{S}$ to K_θ is unitarily equivalent to V and $\varphi_t(\tilde{U}) - \varphi_t(U) \in \mathfrak{S}_p$ for all $p > 1$. Unitary dilations will be considered in detail elsewhere.

The paper is organized as follows. It is shown in §2 that the conditions, imposed on the operator V in the main results, are necessary. Then we consider at first the case where V is a singular unitary operator of multiplicity 1 and we may apply Clark's construction. The model is determined by an inner function θ, or by a singular measure μ on the unit circle connected with θ by relation (4) below, or by a measure ν on the real line associated with μ. In this case we have $rank\,(\tilde{S} - S) = 1$. We obtain conditions on μ, ν, under which the operators $\varphi_t(\tilde{S}) - \varphi_t(S)$ belong to $\mathfrak{S}_2$ and $\mathfrak{S}_1$ for all $t > 0$. We prove Theorem 1.1 in the partial case of multiplicity 1 via properties of the measures μ, ν from our construction, and then from this special case we infer the general case of the theorem. Theorems 1.2 and 1.3 are obtained as corollaries from Theorem 1.1.

The authors are grateful to R.V. Romanov for the useful discussions and for the help with the proof of Theorem 2.1, and also to A.B. Aleksandrov for many helpful suggestions which have improved the exposition.

2. Unitary groups, isometric semigroups, and their cogenerators

In this section we recall some basic properties of generators and cogenerators of strongly continuous unitary groups and their analogs for isometric semigroups. A detailed exposition of these topics may be found in [4, 10, 11].

If $(U_t)_{t\in\mathbb{R}}$ is a strongly continuous unitary group, then its *generator* A is defined by $Ax = \lim_{t\to 0} \frac{U_t - I}{t} x$ on the set of those vectors x for which the limit exists. Then

iA is a selfadjoint operator, and $U_t = \exp(tA)$. The *cogenerator* is the unitary operator $B = (A+I)(A-I)^{-1}$. A necessary and sufficient condition for a unitary operator to be a cogenerator of some unitary group is that its point spectrum does not contain the point 1. For the elements of the group we have the formula expressing them via the cogenerator:

$$U_t = \varphi_t(B) \tag{3}$$

with φ_t defined by (2). Moreover, since

$$\int_0^{+\infty} e^{-t}\varphi_t(z)\,dt = \frac{1-z}{2},$$

we obtain

$$\int_0^{+\infty} e^{-t}U_t\,dt = \int_0^{+\infty} e^{-t}\varphi_t(B)\,dt = \frac{I-B}{2};$$

hence the cogenerator can be expressed via the elements of the group with $t \geq 0$ as

$$B = I - 2\int_0^{+\infty} e^{-t}U_t\,dt.$$

Suppose now that we are given a strongly continuous semigroup $(V_t)_{t\geq 0}$ of isometric operators in a Hilbert space H_0. Then one may consider its unitary dilation, i.e., a unitary group $(U_t)_{t\in\mathbb{R}}$ acting on a wider space H such that H_0 is an invariant subspace for U_t with $t > 0$, and $V_t = U_t|_{H_0}$, $t > 0$ (see [10]). If $t \geq 0$, then $\varphi_t \in H^\infty$. Therefore, H_0 is also an invariant subspace for the cogenerator B of (U_t), and it is natural to define the cogenerator of the isometric semigroup (V_t) as the restriction of B to H_0. Formula (3) holds true if instead of B and U_t, $t > 0$, we apply it to their restrictions to H_0.

For the classes of operators under consideration, the spectral type of an operator is defined as the class of all operators that are unitarily equivalent to the original one. Since unitary groups and isometric semigroups are uniquely determined by their cogenerators and vice versa, it is natural to define the spectral type of a group (semigroup) via the spectral type of its cogenerator. For the class of unitary operators, the spectral type is determined by a scalar measure on the unit circle $\mathbb{T}$, under which the spectral measure of a unitary operator is absolutely continuous, and an integer-valued function on $\mathbb{T}$ counting the local multiplicity of the unitary operator at almost all points relative to the measure. According to the Wold–Kolmogorov decomposition of the cogenerator, every strongly continuous isometric semigroup splits into a direct sum of its unitary and pure isometric parts. The former is a semigroup of unitary operators indexed by $\mathbb{R}_+$, and it may be naturally extended to a unitary group with indices in $\mathbb{R}$. The latter is a semigroup of completely non-unitary operators, i.e., operators unitarily equivalent to a unilateral shift. Thus, the spectral type of an isometric

semigroup is determined by the spectral type of the cogenerator of its unitary part and by the multiplicity of the unilateral shifts.

The following (probably, known) statement shows that the multiplicity of the unilateral shift can not be changed by compact perturbations of the semigroup, while trace class perturbations preserve the absolutely continuous part of the cogenerator. To make the exposition self-contained we present a short proof of this fact. The proof below was communicated to the authors by R.V. Romanov.

Theorem 2.1. *Let $(V_t)_{t\geq 0}$, $(\tilde{V}_t)_{t\geq 0}$ be strongly continuous semigroups of isometric operators in a Hilbert space H.*

1) *If the operator $\tilde{V}_t - V_t$ is compact for all $t \geq 0$, then the pure isometric parts of the semigroups $(V_t)_{t\geq 0}$ and $(\tilde{V}_t)_{t\geq 0}$ are unitarily equivalent, i.e., the multiplicities of the shifts coincide.*

2) *If $\tilde{V}_t - V_t \in \mathfrak{S}_1$ for all $t \geq 0$, then the absolutely continuous parts of the semigroups $(V_t)_{t\geq 0}$ and $(\tilde{V}_t)_{t\geq 0}$ are unitarily equivalent.*

For the proof of the theorem we will need the following lemma.

Lemma 2.2. *Let $T(t)$, $t \geq 0$, be a strongly continuous family of compact operators such that $\sup_{t\geq 0} \|T(t)\| < \infty$. Then the operator $\int_0^\infty e^{-t} T(t) dt$ is compact.*

Proof. For any t consider the Schmidt expansion $T(t) = \sum_{j=1}^\infty s_{tj}(\cdot, x_{tj}) y_{tj}$, where $x(_{tj})$, (y_{tj}) are orthonormal systems and $s_{tj} \searrow 0$. Note that the functions $t \mapsto s_{tj}$ are continuous from below for all j and, therefore, they are measurable. The functions $t \mapsto x_{tj}$ and $t \mapsto y_{tj}$. are also measurable. The norm of the operator $\sum_{j\geq n} s_{tj}(h, x_{tj}) y_{tj}$ equals s_{tn}, and hence

$$\| \int_0^\infty e^{-t} \sum_{j\geq n} s_{tj}(\cdot, x_{tj}) y_{tj}\, dt \| \leq \int_0^\infty e^{-t} s_{tn}\, dt.$$

By the hypothesis, $\sup_{t\geq 0} s_{tn} \leq \sup_{t\geq 0} \|T(t)\| < \infty$. For any t we have $s_{tn} \to 0$, $n \to \infty$, whence, by the Lebesgue theorem, we obtain $\int_0^\infty e^{-t} s_{tn} dt \to 0$, and so $\| \int_0^\infty e^{-t} \sum_{j\geq n} s_{tj}(\cdot, x_{tj}) y_{tj}\, dt \| \to 0$ as $n \to \infty$.

It remains to show that the operator $\int_0^\infty e^{-t} \sum_{j<n} s_{tj}(\cdot, x_{tj}) y_{tj}\, dt$ is compact for all n. To show this, note that the norms of the operators $(\cdot, x_{tj}) y_{tj}$ in any class $\mathfrak{S}_p$ are equal to 1, whence $\mathfrak{S}_p$-norms of the operators $\int_0^\infty e^{-t} \sum_{j=1}^{n-1} s_{tj}(\cdot, x_{tj}) y_{tj}\, dt$ do not exceed

$$\int_0^\infty e^{-t} \sum_{j=1}^{n-1} s_{tj}\, dt \leq (n-1) \cdot \sup_{t\geq 0} \|T(t)\| \int_0^\infty e^{-t} dt = (n-1) \cdot \sup_{t\geq 0} \|T(t)\|.$$

$\square$

Proof of Theorem 2.1. We prove statement 1 of the theorem. Applying Lemma

2.2 to $T(t) = \tilde{V}_t - V_t$, we obtain that the operator $\int_0^\infty e^{-t}(\tilde{V}_t - V_t)dt$ is compact. This means that the difference of the cogenerators of the semigroups (V_t) and $(\tilde{V}_t)$ is compact. Hence, the cogenerators have equal Fredholm indices, which are exactly the multiplicities of the shift.

Now we prove statement 2. Denote by Q the subspace where the cogenerator of the semigroup $(\tilde{V}_t)$ acts as an absolutely continuous unitary operator. Denote by Z the natural embedding of the Q into H. It follows from the assumption $\tilde{V}_t - V_t \in \mathfrak{S}_1$ that $Z(\tilde{V}_t|_Q) - \gamma_t Z \in \mathfrak{S}_1$, where $(\gamma_t)_{t\in\mathbb{R}}$ is the unitary dilation of the semigroup (V_t). Then, by the classical scattering theory for a pair of unitary operators, a strong limit of the isometries $\gamma_t Z(\tilde{V}_t|_Q)^{-1}$ exists as $t \to +\infty$, which defines an isometric wave operator W (see [12, Theorem 6.5.5]). By the construction, the range of the operator W is contained in H and reduces the group (γ_t). Hence, the operator W realizes a unitary equivalence between the restriction of the semigroup $(\tilde{V}_t)$ to Q and some unitary part of the semigroup (V_t).

We have shown that the absolutely continuous unitary part of the semigroup $(\tilde{V}_t)$ is unitary equivalent to some part of the semigroup (V_t). Analogously, the absolutely continuous unitary part of the semigroup (V_t) is unitarily equivalent to some part of the semigroup $(\tilde{V}_t)$. Then, by the spectral theorem, the absolutely continuous unitary parts of the semigroups (V_t) and $(\tilde{V}_t)$ are unitarily equivalent. $\square$

We have an immediate consequence for the perturbations of the semigroup of shifts.

Corollary 2.3. *Let $(\tau_t)_{t\geq 0}$ be the semigroup of shifts and let $(\tilde{\tau}_t)_{t\geq 0}$ be a strongly continuous semigroup of isometric operators in $L^2(\mathbb{R}_+)$.*

1) *If the operator $\tilde{\tau}_t - \tau_t$ is compact for all $t \geq 0$, then there exists a unitary group $(\omega_t)_{t\in\mathbb{R}}$ such that the semigroup $(\tilde{\tau}_t)$ is unitarily equivalent to the direct sum $(\tau_t) \oplus (\omega_t)$, $t \geq 0$.*

2) *If, additionally, $\tilde{\tau}_t - \tau_t \in \mathfrak{S}_1$ for $t \geq 0$, then the spectral measure (of each element or, equivalently, of the cogenerator) of the group (ω_t) is singular.*

For the model on the unit circle Corollary 2.3 means that under the assumption $\varphi_t(\tilde{S}) - \varphi_t(S) \in \mathfrak{S}_1$, $t > 0$, the multiplicity of the shift for the operator $\tilde{S}$ is 1, and the spectral measure of its unitary part is necessarily singular with respect to the Lebesgue measure.

3. The spaces K_θ and the model construction

In this section we introduce a special model of a perturbation satisfying (i)–(iii) starting from a singular measure μ on the unit circle $\mathbb{T}$ and an inner function θ associated with μ.

Let μ be a measure on $\mathbb{T}$ singular relative to the Lebesgue measure m, and let $\mu(\{1\}) = 0$. Define the function θ by

$$\frac{1+\theta(z)}{1-\theta(z)} = \int_{\mathbb{T}} \frac{1+\bar{\xi}z}{1-\bar{\xi}z}\, d\mu(\xi), \qquad z \in \mathbb{D}. \tag{4}$$

It is well known that θ is an inner function. The measure μ is concentrated on the set where angular boundary limits of θ exist and are equal to 1. The measure μ is said to be a Clark measure of the function θ.

For $u \in L^2(\mu)$ put

$$(\Omega u)(z) = (1-\theta(z)) \int_{\mathbb{T}} \frac{u(\xi) d\mu(\xi)}{1-\bar{\xi}z}. \tag{5}$$

Clark [18] proved that Ω is a unitary operator from $L^2(\mu)$ to K_θ. Moreover, angular boundary values of the function Ωu exist and coincide with u μ-almost everywhere [9]. Analogs of these results for the spaces of vector-valued functions may be found in [20].

In this section V is the operator of multiplication by the independent variable ξ on $L^2(\mu)$. Note that 1 is not an eigenvalue for V, since $\mu(\{1\}) = 0$. We find a formula for the unitary operator $\Omega V \Omega^*$, which is a unitarily equivalent transplantation of V to K_θ. For $h = \Omega u$, $u \in L^2(\mu)$, we have

$$\begin{aligned}(\Omega V \Omega^* h)(z) - zh(z) &= (\Omega V u)(z) - z(\Omega u)(z)\\ &= (1-\theta(z)) \int_{\mathbb{T}} \frac{(\xi - z)u(\xi)d\mu(\xi)}{1-\bar{\xi}z}\\ &= (1-\theta(z)) \int_{\mathbb{T}} \xi u(\xi) d\mu(\xi).\end{aligned}$$

Since $\int_{\mathbb{T}} \xi u(\xi) d\mu(\xi) = (u, \bar{\xi})_{L^2(\mu)} = (h, \Omega\bar{\xi})_{K_\theta}$, we obtain

$$\Omega V \Omega^* h = zh + (h, g)(1-\theta), \quad h \in K_\theta,$$

where $g = \Omega\bar{\xi} \in K_\theta$ (it is easy to check the formula for g, $g(z) = \frac{\theta(z)-\theta(0)}{z(1-\theta(0))}$, which will not be used in this paper).

Recall that S denotes the shift operator on H^2, and define the operator $\tilde{S}$ on H^2 by

$$\tilde{S} = S + (\cdot, g)(1-\theta). \tag{6}$$

We have shown that K_θ is an invariant subspace for $\tilde{S}$, and that the restriction of $\tilde{S}$ to K_θ is unitarily equivalent to V. Clearly, $\tilde{S}$ coincides with S on K_θ. We have obtained the following proposition.

Proposition 3.1. *For the operator $\tilde{S}$ defined by* (6) *properties* (i)–(iii) *are fulfilled; the restriction of $\tilde{S}$ to K_θ is unitarily equivalent to the operator V of*

multiplication by the independent variable on the space $L^2(\mu)$. Thus, $\tilde{S}$ is unitarily equivalent to $S \oplus V$.

The operator $\tilde{S}$ differs from the multiplication by z by a rank-one operator: $\tilde{S} - S = (\cdot, g)(1-\theta)$, for the norm of which we have

$$\|\tilde{S} - S\| = \|g\|_{K_\theta} \cdot \|1-\theta\|_{H^2} = \|\bar{\xi}\|_{L^2(\mu)} \cdot \|1-\theta\|_{H^2} < 2\sqrt{\mu(\mathbb{T})}. \tag{7}$$

We shall work mainly with measures μ satisfying an additional condition

$$\int_{\mathbb{T}} \frac{d\mu(\xi)}{|1-\xi|^2} < \infty, \tag{8}$$

This implies, in particular, that $\mu(\{1\}) = 0$. Condition (8) is well known in the theory of inner functions and model subspaces. It is equivalent to any of the following (see, e.g., [22, Chapter VI]):

(i) the function θ defined by (4) has a finite angular derivative at the point 1;

(ii) each function from K_θ has a finite nontangential limit at the point 1.

Moreover, the function $\frac{1-\overline{\theta(1)}\theta}{1-z}$ belongs to K_θ and is the reproducing kernel at the point 1,

$$\left\|\frac{1-\overline{\theta(1)}\theta}{1-z}\right\|_{K_\theta}^2 = \left\|\frac{1-\overline{\theta(1)}}{1-z}\right\|_{L^2(\mu)}^2 < 4\int_{\mathbb{T}} \frac{d\mu(\xi)}{|1-\xi|^2}.$$

4. Equivalent models

Our original construction was a model of shifts in the space $L^2(\mathbb{R}_+)$. We shall also work in other models, which are unitarily equivalent the original one.

An equivalent model construction on the real line may be obtained from the model of shifts by means of the Fourier transform. It sends the space $L^2(\mathbb{R}_+)$ to the Hardy space $H^2(\mathbb{C}_+)$ in the upper half-plane. The isometric semigroup $(\tau_t)_{t\geq 0}$ of shifts becomes the semigroup of operators of multiplication by the functions $\exp(itz)$.

For the model on the unit circle $\mathbb{T}$ we work in the Hardy space H^2, which is a subspace of L^2. In the notations L^2, H^2 we omit the measure, then this means the Lebesgue measure m on $\mathbb{T}$ normalized so that $m(\mathbb{T}) = 1$. The cogenerator of the unperturbed isometric semigroup is the operator S of multiplication by the independent variable z. Then the semigroup consists of operators of the form $\varphi_t(S)$ with φ defined by (2). In other words, our isometric semigroup is the group of operators of multiplication by the inner functions φ_t, $t \geq 0$.

Now we list some formulas establishing a unitary equivalence of the models of multiplications on the unit circle $\mathbb{T}$ and on the real line $\mathbb{R}$. For the variable z on

$\mathbb{T}$ we write $x = i\frac{1+z}{1-z} \in \mathbb{R}$. Given a measure μ on $\mathbb{T}$, define the measure ν on $\mathbb{R}$ by

$$d\mu(z) = \frac{d\nu(x)}{\pi(1+x^2)}. \tag{9}$$

Condition (8) is equivalent to $\nu(\mathbb{R}) < \infty$. The mapping

$$u \mapsto v, \quad v(x) = \frac{1}{\sqrt{\pi}(x+i)} \cdot u\Big(\frac{x-i}{x+i}\Big), \tag{10}$$

is a unitary operator from $L^2(\mu)$ onto $L^2(\nu)$, and also from $L^2 = L^2(\mathbb{T})$ to $L^2(\mathbb{R})$; the latter maps the Hardy class H^2 onto $H^2(\mathbb{C}_+)$. The function $u \in L^2(\mu)$ is expressed via $v \in L^2(\nu)$ as

$$u(z) = \frac{2i\sqrt{\pi}}{1-z} \cdot v\left(i\frac{1+z}{1-z}\right).$$

5. Functions of operators S and $\tilde{S}$

We deal with the construction, where $\tilde{S}$ is a rank-one perturbation of S defined by (6). Take a function $\varphi \in H^\infty$ and suppose that values of φ are defined μ-almost everywhere.

On the subspace θH^2, the operator $\tilde{S}$ coincides with S, hence $\varphi(\tilde{S})$ coincides with $\varphi(S)$ there. Thus, we need to study only the restriction of $\varphi(\tilde{S}) - \varphi(S)$ to K_θ, which makes natural to consider the operator

$$X : L^2(\mu) \to H^2, \quad X = (\varphi(\tilde{S}) - \varphi(S))\Omega, \tag{11}$$

with Ω defined by (5). Take $u \in L^2(\mu)$. Then

$$\begin{aligned}(Xu)(z) &= (1-\theta(z)) \int_{\mathbb{T}} \frac{\varphi(\xi)u(\xi)}{1-\bar{\xi}z} d\mu(\xi) - \varphi(z)(1-\theta(z)) \int_{\mathbb{T}} \frac{u(\xi)}{1-\bar{\xi}z} d\mu(\xi) \\ &= (1-\theta(z)) \int_{\mathbb{T}} \frac{\varphi(\xi)-\varphi(z)}{\xi - z} \xi u(\xi) d\mu(\xi), \qquad u \in L^2(\mu).\end{aligned} \tag{12}$$

Since Ω is a unitary operator from $L^2(\mu)$ onto K_θ, the operator $\varphi(\tilde{S}) - \varphi(S)$ belongs to a class $\mathfrak{S}_p$ if and only if $X \in \mathfrak{S}_p$, and their $\mathfrak{S}_p$-norms coincide.

Equality (10) establishes unitary correspondences between $L^2(\mu)$ and $L^2(\nu)$, and between H^2 and $H^2(\mathbb{C}_+)$. Using these identifications, we construct the

operator $Y : L^2(\nu) \to H^2(\mathbb{C}_+)$ as a unitary transplantation of X. We have

$$\begin{aligned}(Yv)(x) &= \frac{1}{\sqrt{\pi}(x+i)}(Xu)\left(\frac{x-i}{x+i}\right)\\ &= \frac{1}{\sqrt{\pi}(x+i)}\cdot(1-\theta(z))\int_{\mathbb{T}}\frac{\varphi(\xi)-\varphi(z)}{\xi-z}\xi u(\xi)d\mu(\xi)\\ &= (1-\Theta(x))\cdot\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\psi(\zeta)-\psi(x)}{\zeta-x}v(\zeta)d\nu(\zeta), \qquad v\in L^2(\nu),\end{aligned} \tag{13}$$

where

$$\psi(x) = \varphi\left(\frac{x-i}{x+i}\right), \qquad \Theta(x) = \theta\left(\frac{x-i}{x+i}\right), \qquad x\in\mathbb{C}_+.$$

Note that if θ is defined by (4), then for Θ we have the formula

$$\frac{1+\Theta(x)}{1-\Theta(x)} = \frac{1}{\pi i}\int\left(\frac{1}{\zeta-x}-\frac{\zeta}{1+\zeta^2}\right)d\nu(\zeta), \qquad x\in\mathbb{C}_+.$$

The following proposition follows directly from the construction of Y.

Proposition 5.1. *The operators $\varphi(\tilde{S})-\varphi(S)$ and Y belong or not to ideals $\mathfrak{S}_p$ simultaneously, and $\|\varphi(\tilde{S})-\varphi(S)\|_{\mathfrak{S}_p} = \|Y\|_{\mathfrak{S}_p}$.*

6. Estimates for the Hilbert–Schmidt class

Define the integral operator $K : L^2(\nu)\to H^2(\mathbb{C}_+)$ by

$$(Kv)(x) = \frac{1}{2\pi i}\int\frac{\psi(\zeta)-\psi(x)}{\zeta-x}v(\zeta)d\nu(\zeta), \qquad v\in L^2(\nu). \tag{14}$$

If $K\in\mathfrak{S}_p$, then $Y\in\mathfrak{S}_p$ as well, and we obviously have

$$\|Y\|_{\mathfrak{S}_p}\le 2\cdot\|K\|_{\mathfrak{S}_p}. \tag{15}$$

For the norm of K in $\mathfrak{S}_2$ we obtain

$$\|K\|^2_{\mathfrak{S}_2} = \frac{1}{4\pi^2}\iint_{\mathbb{R}\times\mathbb{R}}\left|\frac{\psi(\zeta)-\psi(x)}{\zeta-x}\right|^2 dx\,d\nu(\zeta). \tag{16}$$

Consider the semigroups of the form $(\varphi_t(S))$, $(\varphi_t(\tilde{S}))$, where $\varphi_t(z) = \exp(t\frac{z+1}{z-1})$, $t>0$. The corresponding functions ψ_t have the form

$$\psi_t(x) = \varphi_t\left(\frac{x-i}{x+i}\right) = e^{itx}.$$

Now we state a condition on ν ensuring that the operator K belongs to the Hilbert–Schmidt class for all t at once. To this end, we shall obtain a precise formula for the integral in (16) with $\psi=\psi_t$; it will give us a sufficient condition

for $\varphi_t(\tilde{S}) - \varphi_t(S) \in \mathfrak{S}_2$ for all t. At the end of the section we present an example of the construction, for which $\varphi_t(\tilde{S}) - \varphi_t(S)$ is not in the Hilbert-Schmidt class for any $t \neq 0$.

Proposition 6.1. *Assume that* (8) *is fulfilled. Then for any $t > 0$ we have*

$$\iint_{\mathbb{R}\times\mathbb{R}} \left|\frac{\psi_t(\zeta) - \psi_t(x)}{\zeta - x}\right|^2 dx\, d\nu(\zeta) = 2\pi t \cdot \nu(\mathbb{R}) = 8\pi^2 t \int_{\mathbb{T}} \frac{d\mu(\xi)}{|1-\xi|^2} < \infty. \tag{17}$$

Proof. For the integral on the left-hand side with respect to the Lebesgue measure, we obtain

$$\int_{\mathbb{R}} \frac{|e^{it\zeta} - e^{itx}|^2}{(\zeta - x)^2}\, dx = 4 \int_{\mathbb{R}} \frac{\sin^2 \frac{t}{2}(\zeta - x)}{(\zeta - x)^2}\, dx = 2\pi t,$$

where we used the well-known formula $\int_{\mathbb{R}} \frac{\sin^2 \alpha s}{s^2} ds = \pi|\alpha|$. The right identity in (17) follows from the relation $\nu(\mathbb{R}) = 4\pi \int \frac{d\mu(\xi)}{|1-\xi|^2}$. $\square$

Corollary 6.2. *If μ satisfies* (8)*, then $\varphi_t(\tilde{S}) - \varphi_t(S) \in \mathfrak{S}_2$ for all $t > 0$;*

$$\|\varphi_t(\tilde{S}) - \varphi_t(S)\|_{\mathfrak{S}_2} \leq 2\sqrt{2t} \left(\int_{\mathbb{T}} \frac{d\mu(\xi)}{|1-\xi|^2}\right)^{1/2}. \tag{18}$$

Proof. This directly follows from Proposition 5.1 and relations (15), (16), and (17). $\square$

In §7 we find a certain "smallness" condition for a measure μ at the point 1, see (21), which is sufficient for the inclusion of the operator $\varphi_t(\tilde{S}) - \varphi_t(S)$ into the trace class $\mathfrak{S}_1$.

From the representation (13) of Y in the form of an integral operator it follows that $Y \in \mathfrak{S}_2$ if and only if

$$\iint_{\mathbb{R}\times\mathbb{R}} |1 - \Theta(z)|^2 \left|\frac{\psi(\zeta) - \psi(x)}{\zeta - x}\right|^2 dx d\nu(\zeta) < \infty.$$

We expect that the condition $\nu(\mathbb{R}) < \infty$ equivalent to (8), is also necessary for the inclusions $\varphi_t(\tilde{S}) - \varphi_t(S) \in \mathfrak{S}_2$, that is, the factor $1 - \Theta(z)$ does not change essentially the convergence of the integral in (16).

Now we present an example of a measure ν such that the corresponding operator $\varphi_t(\tilde{S}) - \varphi_t(S)$ is not in the Hilbert–Schmidt class for any $t \neq 0$. Recall that $\tilde{S} - S$ is a rank-one operator whose norm can be made arbitrarily small.

Example 6.3. Let $\nu = \sum_{n\in\mathbb{Z}} \delta_n$ be the sum of unit point masses at all integers. Thus, the total mass of ν is infinite, and hence the corresponding measure μ on

$\mathbb{T}$ does not satisfy condition (8). For $\psi_t(x) = e^{itx}$ in the last formula, we obtain the integral

$$\int_{\mathbb{R}} |1-\Theta(s)|^2 \left(\sum_{n\in\mathbb{Z}} \frac{\sin^2 \frac{t}{2}(s-n)}{(s-n)^2} \right).$$

Obviously, for any $t \neq 0$, there exist positive constants $\delta(t) < 1/2$ and $C(t)$ such that the sum in brackets in the above formula is bounded from below by $C(t)$ for $s \in (k+\delta(t), k+2\delta(t))$, $k \in \mathbb{Z}$. On the other hand, an elementary estimate shows that for $s \in (k+\delta(t), k+2\delta(t))$, $k \in \mathbb{Z}$,

$$\left| \frac{1+\Theta(s)}{1-\Theta(s)} \right| = \left| \sum_{n\in\mathbb{Z}} \frac{1+sn}{(s-n)(n^2+1)} \right| \le C_1(t),$$

and therefore $|1-\Theta(s)| \ge C_2(t) > 0$. Combining these estimates, we conclude that the integral diverges, i.e., Y does not belong to the Hilbert–Schmidt class.

7. Estimates for the trace class

In this section, instead of the Hilbert–Schmidt class $\mathfrak{S}_2$, we consider the classes $\mathfrak{S}_p$ with $p < 2$, and, in the first place, the trace class $\mathfrak{S}_1$. According to (15), to show that $\varphi_t(\tilde{S}) - \varphi_t(S) \in \mathfrak{S}_p$, we need to prove that $K \in \mathfrak{S}_p$, where K is defined by (14) with $\psi = \psi_t$, $\psi_t(x) = e^{itx}$. We reduce this problem to a question about embeddings of the Paley–Wiener space $\mathcal{PW}_t$ of all entire functions of exponential type at most t that are square-summable on $\mathbb{R}$. Embedding operators for spaces of analytic functions and their inclusion to ideals $\mathfrak{S}_p$ were studied in detail by O.G. Parfenov [7, 8]. In [3] generalizations of some of Parfenov's results to embeddings of coinvariant subspaces K_θ are obtained.

Consider the operator adjoint to the operator K defined by (14) with $\psi = \psi_t$, $\psi_t(x) = e^{itx}$:

$$\begin{aligned}(K^*f)(\zeta) &= -\frac{1}{2\pi i} \int \frac{e^{-it\zeta} - e^{-itx}}{\zeta - x} f(x)dx \\ &= e^{-\frac{it\zeta}{2}} \int e^{-\frac{itx}{2}} f(x) \left(\frac{1}{\pi} \cdot \frac{\sin \frac{t}{2}(x-\zeta)}{x-\zeta} \right) dx, \quad f \in L^2(\mathbb{R}).\end{aligned}$$

It is well known that the function $\frac{1}{\pi} \cdot \frac{\sin \frac{t}{2}(x-\zeta)}{x-\zeta}$ is the reproducing kernel at ζ for the space $\mathcal{PW}_{t/2}$, which means that this function belongs to $\mathcal{PW}_{t/2}$, and the inner product of any $f \in \mathcal{PW}_{t/2}$ with the reproducing kernel equals $f(\zeta)$. Therefore, $K^*f = 0$ if the function $\tilde{f}$, $\tilde{f}(x) = e^{-\frac{itx}{2}} f(x)$, is orthogonal to $\mathcal{PW}_{t/2}$, and if $\tilde{f} \in \mathcal{PW}_{t/2}$, then

$$(K^*f)(\zeta) = e^{-\frac{it\zeta}{2}} \cdot \tilde{f}(\zeta) = e^{-it\zeta} f(\zeta).$$

Thus, K^* belongs to a class $\mathfrak{S}_p$ simultaneously with the operator $E_{\nu,t/2}$ which embeds the space $\mathcal{PW}_{t/2}$ into $L^2(\nu)$.

The following criterion is due to Parfenov [7, 8].

Theorem 7.1. *Let $\Delta_n = [n, n+1)$. Then $E_{\nu,t} \in \mathfrak{S}_p$, $p > 0$, if and only if*

$$\sum_{n\in\mathbb{Z}} \big(\nu(\Delta_n)\big)^{p/2} < \infty. \tag{19}$$

Moreover,

$$\|E_{\nu,t}\|^p_{\mathfrak{S}_p} \le C(p)\, t^{p/2} \sum_{n\in\mathbb{Z}} \big(\nu(\Delta_n)\big)^{p/2}. \tag{20}$$

Going back to the measure μ, condition (19) reads as

$$\sum_{n\in\mathbb{Z}} \bigg(\int_{\gamma_n} \frac{d\mu(\xi)}{|1-\xi|^2} \bigg)^{p/2} < \infty,$$

where the arcs γ_n are defined for $n > 0$ by $\gamma_n = \{e^{i\varphi} : \pi/(n+1) \le \varphi \le \pi/n\}$ and symmetrically for $n < 0$.

We obtain a condition, which is analogous to (8) and is sufficient for (19) and, therefore, sufficient for the inclusion of $\varphi_t(\tilde S) - \varphi_t(S)$ into the class $\mathfrak{S}_p$.

Proposition 7.2. *Let $0 < p < 2$. If the measure μ satisfies*

$$\int_{\mathbb{T}} \frac{d\mu(\xi)}{|1-\xi|^q} < \infty \tag{21}$$

for some $q > 1 + 2/p$, then $K \in \mathfrak{S}_p$, and, consequently, $\varphi_t(\tilde S) - \varphi_t(S) \in \mathfrak{S}_p$.

In particular, (21) with $q > 3$ yields $\varphi_t(\tilde S) - \varphi_t(S) \in \mathfrak{S}_1$ and

$$\|\varphi_t(\tilde S) - \varphi_t(S)\|_{\mathfrak{S}_1} \le M_q \cdot t^{1/2} \cdot \bigg(\int_{\mathbb{T}} \frac{d\mu(\xi)}{|1-\xi|^{\,q}} \bigg)^{1/2}, \tag{22}$$

where M_q is a constant depending only on q.

Proof. Rewrite condition (21) in terms of the measure ν defined by (9):

$$\int_{\mathbb{R}} (x^2+1)^{r/2} d\nu(x) = 2^{-q}\pi \int_{\mathbb{T}} \frac{d\mu(\xi)}{|1-\xi|^q} < \infty, \tag{23}$$

where $r = q - 2 > (2-p)/p$. Then, by the Hölder inequality with exponents $2/p$ and $2/(2-p)$,

$$\begin{aligned} \bigg(\sum_{n\in\mathbb{Z}} \big(\nu(\Delta_n)\big)^{p/2} \bigg)^{2/p} &\le \sum_{n\in\mathbb{Z}} (|n|+1)^r\, \nu(\Delta_n) \cdot \bigg(\sum_{n\in\mathbb{Z}} (|n|+1)^{-pr/(2-p)} \bigg)^{(2-p)/p} \\ &= \mathrm{const} \cdot \sum_{n\in\mathbb{Z}} (|n|+1)^r\, \nu(\Delta_n) \le \mathrm{const} \cdot \int_{\mathbb{R}} (|t|+1)^r d\nu(t). \end{aligned}$$

Now the statement follows from Parfenov's theorem and inequality (20). $\square$

Example 7.3. The exponent 3 is sharp, and for $q = 3$ the second statement of Proposition 7.2 fails. Indeed, choose ν so that $\nu(\Delta_n) = \big((|n|+1)\log(|n|+2)\big)^{-2}$. Then (23) is fulfilled, but (19) is not, and hence K is not in $\mathfrak{S}_1$.

8. Operators of multiplicity > 1

Now we use the construction developed above for the case of multiplicity 1, for a generalization to the operators with arbitrary multiplicity.

Let $\{\mu_n\}$ be a family of singular measures on the unit circle, where n runs over the set $n = 1, 2, \dots, N$ for some positive integer N or over $\mathbb{N}$. We assume that for some $q > 3$

$$\sum_n \left(\int_{\mathbb{T}} \frac{d\mu_n(\xi)}{|1-\xi|^q} \right)^{1/2} < \infty. \tag{24}$$

For each n, we construct objects as in §3: the function θ_n is determined by formula (4) with μ_n in place of μ, the operator $\Omega_n : L^2(\mu_n) \to K_{\theta_n}$ acts by formula (5) with $g_n = \Omega_n\bar{\xi}$, V_n is the operator of multiplication by z on $L^2(\mu_n)$. Set

$$\hat{\theta}_n = \prod_{k=1}^{n-1} \theta_k.$$

The operator $\Omega : \sum \oplus L^2(\mu_n) \to H^2$ will be defined here by

$$\Omega\Big(\sum \oplus u_n\Big) = \sum \hat{\theta}_n \Omega_n u_n.$$

Since condition (24) is fulfilled for some $q > 3$, it is also true for $q = 2$. Hence,

$$\sum_n \left\| \frac{1-\overline{\theta_n(1)}\theta_n}{1-z} \right\|_{L^2} < 2 \sum_n \left(\int_{\mathbb{T}} \frac{d\mu_n(\xi)}{|1-\xi|^2} \right)^{1/2} < \infty. \tag{25}$$

Therefore, the series

$$\sum_n \overline{\hat{\theta}_n(1)}\hat{\theta}_n \frac{(1-\overline{\theta_n(1)}\theta_n)}{1-z}$$

converges in the norm of the space L^2. The partial sums have the form $\frac{1-\overline{\hat{\theta}_n(1)}\hat{\theta}_n}{1-z}$. Hence the limit function can be written as $\frac{1-\theta}{1-z}$ for an inner function θ that coincides with the product of the functions θ_n up to a unimodular multiplicative constant. It is easily seen that Ω isometrically maps $\sum \oplus L^2(\mu_n)$ onto K_θ.

Define the operator $\tilde{S}$ on H^2 by

$$\tilde{S} = S + \sum_n (\cdot, \hat{\theta}_n g_n)\, \hat{\theta}_n \cdot (1 - \theta_n), \tag{26}$$

which is a generalization of (6). Then the operator $\tilde{S}$ is diagonal with respect to the decomposition $H^2 = \sum \oplus \hat{\theta}_n K_{\theta_n} \oplus \theta H^2$. Indeed, $\tilde{S}$ coincides with S on θH^2, while on each subspace $\hat{\theta}_n K_{\theta_n}$, $\tilde{S}$ is a unitary transplantation of multiplication by z on $L^2(\mu_n)$. By a direct computation, we obtain $\tilde{S}(\hat{\theta}_n f) = \hat{\theta}_n \Omega_n(zu) \in \hat{\theta}_n K_{\theta_n}$ for $f = \Omega_n u \in K_{\theta_n}$.

Thus, $\tilde{S}$ satisfies properties (i)–(iii). Similarly to (7), we obtain the estimate for the trace class norm of $\tilde{S} - S$:

$$\|\tilde{S} - S\|_{\mathfrak{S}_1} < 2 \sum_n \sqrt{\mu_n(\mathbb{T})}. \tag{27}$$

To estimate the trace class norm of $\varphi_t(\tilde{S}) - \varphi_t(S)$, we use inequality (22), which gives us

$$\|\varphi_t(\tilde{S}) - \varphi_t(S)\|_{\mathfrak{S}_1} \le M_q \cdot t^{1/2} \cdot \sum_n \left(\int_{\mathbb{T}} \frac{d\mu_n(\xi)}{|1 - \xi|^{\,q}} \right)^{1/2}. \tag{28}$$

Proof of Theorem 1.1. Take an arbitrary unitary operator V whose spectral measure is singular relative to the Lebesgue measure and has no point mass at 1. Then there exist singular unitary operators V_n of multiplicity 1 such that $V = \oplus \sum V_n$. Construct measures μ_n so that each operator V_n is unitarily equivalent to multiplication by the independent variable ξ on $L^2(\mu_n)$. We may think the measures μ_n satisfy (24) with $q > 3$ and $\sum_n \sqrt{\mu_n(\mathbb{T})} < \varepsilon/2$; otherwise, these properties may be fulfilled after multiplying the measures μ_n by appropriate positive weights. Consider the operator $\tilde{S}$ defined by (26). Then (27) yields $\|\tilde{S} - S\|_{\mathfrak{S}_1} < \varepsilon$, and for any $t > 0$ the operator $\varphi_t(\tilde{S}) - \varphi_t(S)$ is of trace class with norm estimated by (28). $\square$

Remark 8.1. For the Hilbert–Schmidt norm, from (18) we obtain the estimate

$$\|\varphi_t(\tilde{S}) - \varphi_t(S)\|_{\mathfrak{S}_2} \le 2\sqrt{2t} \left(\sum_n \int_{\mathbb{T}} \frac{d\mu_n(\xi)}{|1 - \xi|^2} \right)^{1/2}.$$

Proof of Theorem 1.2. Statement 1 is a particular case of Theorem 2.1. Statement 2 follows immediately from Theorem 1.1 when we pass to the model connected with the group of shifts on the line. Indeed, in the construction in Theorem 1.1 the differences of the elements of the semigroups on $L^2(\mathbb{R}_+)$ belong to the trace class, as required. $\square$

Proof of Theorem 1.3. Let V be an arbitrary unitary operator such that 1 is not its eigenvalue. Then $V = (A - iI)(A + iI)^{-1}$ for some selfadjoint operator A, and $\varphi_t(V) = \exp(itA)$. Applying a variant of the Weil–von Neumann theorem which is due to Kuroda (see [5] and [12, Theorem 6.2.5]) to some cross-normed ideal which is contained in all ideals $\mathfrak{S}_p$ with $p > 1$, we may construct a close selfadjoint operator with pure point spectrum for a given selfadjoint operator so that the difference between the perturbed operator and the original one belongs to $\mathfrak{S}_p$ for all $p > 1$, and the norms may be taken to be arbitrarily small.

We represent the operator A as the direct sum of bounded operators A_n and, applying the Kuroda theorem, construct operators A'_n, so that the norms $\|A_n - A'_n\|_{\mathfrak{S}_p}$ are small. It follows from the results by Davies [19] that $\exp(itA_n) - \exp(itA'_n)$ belongs to all classes $\mathfrak{S}_p$, $p > 1$, for $0 \le t \le 1$. Thus, we may construct a selfadjoint operator A' such that $\exp(itA) - \exp(itA')$ is in all classes $\mathfrak{S}_p$, $p > 1$, for $0 \le t \le 1$. Hence this holds for any $t > 0$, since the inclusion $\exp(itA) - \exp(itA') \in \mathfrak{S}_p$ implies the inclusion $\exp(2itA) - \exp(2itA') \in \mathfrak{S}_p$. Now the statement follows from Theorem 1.1 applied to the unitary operator $V' = (A' - iI)(A' + iI)^{-1}$. $\square$

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